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ON SPECTRAL ANALYSIS OF NON-MONIC MATRIX AND OPERATOR POLYNOMIALS, II. DEPENDENCE ON THE FINITE SPECTRAL DATA

BY

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ABSTRACT

This paper studies the dependence of matrix polynomials on the finite spectral data, i.e., the spectrum at infinity is not taken into account. A description and classification of right divisors based only on the finite part of the spectrum is also given.

Introduction

This paper is a continuation of the earlier paper [3] and is based on the results obtained there. In [3], matrix polynomials were studied via the complete spectral data (including the spectral data at infinity). This paper studies the influence of the spectral structure at finite points only.

The main definitions and notations from [3] are maintained in this paper. Some of them are reviewed briefly for the convenience of the reader in the next section.

§1. Preliminaries

Let $L(\lambda) = I + \Sigma_{j=1}^{l} \lambda^j A_j$ be a comonic matrix polynomial with $n \times n$ matrix coefficients A_j. An admissible pair of matrices (X_F, J_F^{-1}) with $n \times p$ matrix X_F and $p \times p$ invertible Jordan matrix J_F, is called a finite canonical pair for $L(\lambda)$, if the following conditions hold:

a) p is the degree of $\det L(\lambda)$;

b) $X_F J_F^{-l} + A_1 X_F J_F^{-l+1} + \cdots + A_l X_F = 0$;

c) rank $\operatorname{col}(X_F J_F^{-j})_{j=0}^{l-1} = p$,

where, as usual, $\operatorname{col}(X_F J_F^{-j})_{j=0}^{l-1}$ denotes the column matrix

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$$\begin{bmatrix} X_F \\ X_F J_F^{-1} \\ \cdots\cdots \\ X_F J_F^{-l+1} \end{bmatrix}$$

In view of theorem 1.1 of [4] and theorem 3.1 of [3], every finite part of a canonical pair for $L(\lambda)$ (as defined in [3]) is a finite canonical pair for $L(\lambda)$ (as defined here). The converse is also true (as one can see from the results of §4 of [3]), so the use of the same notation (X_F, J_F^{-1}) for both things is justified.

We shall also use the following definitions and notations: a $q \times p$ matrix A^+ is said to be a generalized inverse for a given $p \times q$ matrix A, if A^+ satisfies the equations $AA^+A = A$, and $A^+AA^+ = A^+$. Note that for any direct complement Q_1 of $\operatorname{Ker} A$ in $\mathbf{C}_q$ and any direct complement Q_2 of $\operatorname{Im} A$ in $\mathbf{C}_p$, there exists a generalized inverse A^+ such that $\operatorname{Im} A^+ = Q$, and $\operatorname{Ker} A^+ = Q_2$. An admissible pair of matrices (X, T) is called *l-independent* if $\operatorname{Ker}[\operatorname{col}(XT^{j-1})_{j=1}^l] = \{0\}$. If $W_1, \cdots, W_l$ are subspaces in $\mathbf{C}_n$, then $\operatorname{col}(W_j)_{j=1}^l$ will denote the subspace in $\mathbf{C}_{nl}$ consisting of $x = \operatorname{col}(x_j)_{j=1}^l$ with $x_j \in W_j$ for $j = 1, 2, \cdots, l$.

§2. Comonic polynomials with a given finite canonical pair

We begin with a preliminary lemma.

LEMMA 2.1. *Let (X, T) be an l-independent admissible pair with an invertible matrix T. Then there exists a sequence of subspaces $\mathbf{C}_n \supseteq W_1 \supseteq \cdots \supseteq W_l$ such that*

$$\operatorname{col}(W_j)_{j=1}^l \dot{+} \operatorname{Im}(\operatorname{col}(XT^{j-1})_{j=1}^l) = \mathbf{C}_{nl}. \tag{2.1}$$

PROOF. We shall construct a sequence of subspaces $\mathbf{C}_n \supseteq W_1 \supseteq \cdots \supseteq W_l$ such that

$$\operatorname{col}(W_j)_{j=k+1}^l \dot{+} \operatorname{Im}\operatorname{col}(XT^{j-1})_{j=k+1}^l = \mathbf{C}_{n(l-k)} \tag{2.2}$$

for $k = 0, \cdots, l-1$.

Let W_1 be a direct complement of $\operatorname{Im} XT^{l-1} = \operatorname{Im} X$ in $\mathbf{C}_n$. Then (2.2) holds for $k = l-1$. Suppose that $W_{i+1} \supseteq \cdots \supseteq W_l$ are already constructed so that (2.2) holds for $k = i, \cdots, l-1$. It is then easy to check that $\operatorname{col}(Z_j)_{j=i}^l \cap \operatorname{Im}\operatorname{col}(XT^{j-1})_{j=i}^l = \{0\}$, where $Z_k = W_k$ for $k = i+1, \cdots, l$, and $Z_i = W_{i+1}$. Hence, the sum

$$S = \operatorname{col}(Z_j)_{j=1}^l \dot{+} \operatorname{Im}\operatorname{col}(XT^{j-1})_{j=i}^l$$

is a direct sum. Let Q be a direct complement of $\operatorname{Ker} A$ in $\mathbf{C}_{n(l-i)}$, where $A = \operatorname{col}(XT^{j-1})_{j=i+1}^{l}$. Let A^{+} be a generalized inverse of A such that $\operatorname{Im} A^{+} = Q$ and $\operatorname{Ker} A^{+} = \operatorname{col}(W_j)_{j=i+1}^{l}$. Let P be the projector on $\operatorname{Ker} A$ along Q. Thus

$$S = \left\{ \begin{pmatrix} y + XT^{i-1}Pz + XT^{i-1}A^{+}x \\ x \end{pmatrix} \,\middle|\, y \in W_{i+1}, x \in \mathbf{C}_{n(l-i)}, z \in \mathbf{C}_r \right\}, \tag{2.3}$$

where r is the *order* of the admissible pair (X, T), i.e. the size of T.

Indeed, if $x \in \mathbf{C}_{n(l-i)}$, then by the assumption of induction, $x = Az_1 + x_1$, where $z_1 \in Q$ and $x_1 \in \operatorname{col}(W_j)_{j=i+1}^{l}$. Hence

$$\begin{pmatrix} XT^{i-1}A^{+}x \\ x \end{pmatrix} = \begin{pmatrix} XT^{i-1}A^{+}(Az_1 + x_1) \\ Az_1 + x_1 \end{pmatrix} = \begin{pmatrix} XT^{i-1}z_1 \\ Az_1 \end{pmatrix} + \begin{pmatrix} 0 \\ x_1 \end{pmatrix}.$$

From the definition of S it follows that

$$\begin{pmatrix} XT^{i-1}A^{+}x \\ x \end{pmatrix} \in S.$$

For any $z \in \mathbf{C}_r$, also

$$\begin{pmatrix} XT^{i-1}Pz \\ 0 \end{pmatrix} = \begin{pmatrix} XT^{i-1}Pz \\ APz \end{pmatrix} \in S,$$

and clearly

$$\begin{pmatrix} y \\ 0 \end{pmatrix} \in S$$

for any $y \in W_{i+1}$. The inclusion $\supseteq$ in (2.3) thus follows. To check the converse inclusion, take $y \in W_{i+1}$, $x_1 \in \operatorname{col}(W_j)_{j=i+1}^{l}$ and $z \in \mathbf{C}_r$. Then

$$\begin{aligned} \begin{pmatrix} y \\ x_1 \end{pmatrix} + \begin{pmatrix} XT^{i-1} \\ A \end{pmatrix} z &= \begin{pmatrix} y \\ x_1 \end{pmatrix} + \begin{pmatrix} XT^{i-1} \\ A \end{pmatrix} [Pz + (I-P)z] \\ &= \begin{pmatrix} y \\ x_1 \end{pmatrix} + \begin{pmatrix} XT^{i-1}Pz \\ 0 \end{pmatrix} + \begin{pmatrix} XT^{i-1}(I-P)z \\ A(I-P)z \end{pmatrix} \\ &= \begin{pmatrix} y \\ 0 \end{pmatrix} + \begin{pmatrix} XT^{i-1}Pz \\ 0 \end{pmatrix} + \begin{pmatrix} XT^{i-1}A^{+}(x_1 + A(I-P)z) \\ x_1 + A(I-P)z \end{pmatrix} \end{aligned}$$

(the last equality follows from $A^{+}x_1 = 0$ and $I - P = A^{+}A$), and (2.3) is proved.

Now, let Y be a direct complement of $W_{i+1} + \operatorname{Im} XT^{i-1}P$ in $\mathbf{C}_n$. Then from (2.3) we obtain

$$S \dotplus \begin{pmatrix} Y \\ 0 \end{pmatrix} = \mathbf{C}_{n(l-i+1)},$$

and so we can put $W_i = W_{i+1} \dotplus Y$. Q.E.D.

Note that $\dim W_i = k_{i-1}$ for $i = 1, \cdots, l$, where k_i are the so-called k-indices of (X, T). We remind the reader of the definitions of k- and of the *s-indices* of (X, T), which were introduced in [4]: $k_i = n + q_{l-i-1} - q_{l-i}$ for $i = 0, \cdots, l-1$, where $q_i = \operatorname{rank} \operatorname{col}(XT^j)_{j=0}^{i-1}$ for $i \geqq 1$ and $q_0 = 0$; s_i for $i = 0, 1, \cdots$ is the number of integers $k_0, k_1, \cdots, k_{l-1}$ which are larger than i.

Lemma 2.1 leads to the following definition. Let (X, T) be an l-independent admissible pair. A left inverse V of $\operatorname{col}(XT^{j-1})_{j=1}^{l}$ is called *special* if $V \mid \operatorname{col}(W_j)_{j=1}^{l} = 0$, where the sequence of subspaces $\mathbf{C}_n \supseteq W_1 \supseteq \cdots \supseteq W_l$ is taken from Lemma 2.1.

Now, let (X_F, J_F^{-1}) be an l-independent admissible pair of order r such that J_F^{-1} is the inverse of a Jordan matrix J_F. The following theorem gives a construction of a comonic polynomial such that (X_F, J_F^{-1}) is its finite canonical pair.

THEOREM 2.1. *The minimal degree of a comonic polynomial $L(\lambda)$ such that (X_F, J_F^{-1}) is its finite canonical pair, is equal to the minimal l such that (X_F, J_F^{-1}) is l-independent. One such polynomial is given by the formula*

$$L_0(\lambda) = I - X_F J_F^{-l}(V_1 \lambda^l + \cdots + V_l \lambda),$$

where $V = (V_1, \cdots, V_l)$ is a special left inverse of $\operatorname{col}(X_F J_F^{-j})_{j=0}^{l-1}$. The infinite part (X_∞, J_∞) of a canonical pair for $L_0(\lambda)$ has the form

$$X_\infty = (x_0 0 \cdots 0\, x_1 0 \cdots 0 \cdots x_\nu 0 \cdots 0),$$

$$J_\infty = \operatorname{diag}(J_{\infty 0}, J_{\infty 1}, \cdots, J_{\infty \nu}),$$

where $J_{\infty j}$ is a nilpotent Jordan block of size s_j, *and $s_0 \geqq \cdots \geqq s_\nu$ are the s-indices of (X_F, J_F^{-1}).*

PROOF. Let (X_F, J_F^{-1}) be a finite canonical pair of some comonic polynomial $L(\lambda)$. Then (see theorem 3.1 [3]) (X_F, J_F^{-1}) is a part of a standard pair for the monic polynomial $\lambda^m L(\lambda^{-1})$, where m is the degree of $L(\lambda)$. Therefore, (X_F, J_F^{-1}) is m-independent, and the first assertion of the theorem follows.

We now prove that $L_0(\lambda)$ has (X_F, J_F^{-1}) as its finite canonical pair. Let $\tilde{L}(\lambda) = \lambda^l L_0(\lambda^{-1}) = \lambda^l I - X_F J_F^{-l}(V_1 + \cdots + V_l \lambda^{l-1})$. In view of theorem 3.1 [3], it is sufficient to prove that (X_F, J_F^{-1}) is a part of a standard pair for $\tilde{L}(\lambda)$. But this was already proved in lemma 7.1 [4].

In order to study the structure of (X_∞, J_∞), it is convenient to consider the companion matrix

$$C = \begin{bmatrix} 0 & I & 0 & \cdots & 0 \\ 0 & 0 & I & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & I \\ X_F J_F^{-l} V_1 & X_F J_F^{-l} V_2 & & \cdots & X_F J_F^{-l} V_l \end{bmatrix}$$

of $\tilde{L}(\lambda)$. The subspace $W = \operatorname{col}(W_j)_{j=1}^{l}$ is C-invariant, because $CW = \operatorname{col}(W_{j+1})_{j=1}^{l} \subset W$ (by definition $W_{l+1} = \{0\}$). The equality $C \cdot [\operatorname{col}(X_F J_F^{-j})_{j=0}^{l-1}] = [\operatorname{col}(X_F J_F^{-j})_{j=0}^{l-1}] \cdot J_F^{-1}$ shows that the subspace $\Theta = \operatorname{Im} \operatorname{col}(X_F J_F^{-j})_{j=0}^{l-1}$ is also C-invariant. It is clear that $C \mid W$ is nilpotent, and $C \mid \Theta$ is invertible. Let $k_0 \geqq k_1 \geqq \cdots \geqq k_{l-1}$ be the k-indices of (X_F, J_F^{-1}). Let $z_1^{(i)}, \cdots, z_{k_{i-1}-k_i}^{(i)}$ be a basis in W_i modulo W_{i+1}, for $i = 1, \cdots, l$ (by definition $k_l = 0$). Then it is easy to see that for $j = 1, \cdots, k_{i-1} - k_i$,

$$\operatorname{col}(\delta_{1p} z_j^{(i)})_{p=1}^{l},\ \operatorname{col}(\delta_{2p} z_j^{(i)})_{p=1}^{l}, \cdots, \operatorname{col}(\delta_{ip} z_j^{(i)})_{p=1}^{l}$$

is a Jordan chain of C corresponding to the eigenvalue 0, of length i. Taking into account the definition of the s-indices and the connection between the Jordan chains of C and the Jordan chains of $L(\lambda)$ (see for example section 2 in [2]), we see that (X_∞, J_∞) has the structure described in the theorem.

The following corollary gives a description of all the comonic matrix polynomials for which (X_F, J_F^{-1}) is a finite canonical pair.

Corollary 2.1. *A comonic matrix polynomial $L(\lambda)$ has (X_F, J_F^{-1}) as its finite canonical pair if and only if $L(\lambda)$ admits the following representation:*

$$L(\lambda) = U(\lambda) \cdot L_0(\lambda),$$

where $L_0(\lambda) = I - X_F J_F^{-l}(V_1 \lambda^l + \cdots + V_l \lambda)$, $(V_1 \cdots V_l)$ is a special left inverse of $\operatorname{col}(X_F J_F^{-j})_{j=1}^{l}$, and $U(\lambda)$ is a comonic matrix polynomial with constant determinant.

That is an immediate corollary of Theorem 2.1 and theorem 5.1 of [3].

§3. Matrix polynomials of minimal degree

In the preceding section, we solved the problem of reconstructing a comonic polynomial, given its finite canonical pair (X_F, J_F^{-1}). Clearly, there are many comonic polynomials of minimal degree with the same (X_F, J_F^{-1}), which differ by their spectral data at infinity. In this section, we give a description of all possible Jordan structures at infinity for such polynomials.

We remind the reader that the Jordan structure at infinity of a comonic polynomial $L(\lambda)$ is described by the *infinite canonical pair* (X_∞, J_∞) of $L(\lambda)$, where the columns of X_∞ form a canonical set of eigenvectors and generalized eigenvectors at infinity for $\lambda^{-l}L(\lambda)$ (l is the degree of $L(\lambda)$), and J_∞ is the corresponding nilpotent Jordan matrix.

THEOREM 3.1. *Let (X_F, J_F^{-1}) be an admissible pair of order r with invertible Jordan matrix J_F. Let l be the smallest positive integer such that (X_F, J_F^{-1}) is l-independent. If (X_F, J_F^{-1}) is a finite canonical pair of a comonic matrix polynomial $L(\lambda)$ of degree l, and (X_∞, J_∞) is its infinite canonical pair, then the sizes $p_1 \geqq p_2 \geqq \cdots \geqq p_\nu$ of the Jordan blocks in J_∞ satisfy the following conditions*:

$$\sum_{i=1}^{j} p_i \geqq \sum_{i=1}^{j} s_{i-1} \quad \textit{for} \quad j = 1, 2, \cdots, \nu; \tag{3.1}$$

$$\sum_{i=1}^{\nu} p_i = nl - r, \tag{3.2}$$

where $s_0, s_1, \cdots$, are the s-indices of (X_F, J_F^{-1}).

Conversely, if for some positive integers $p_1 \geqq p_2 \geqq \cdots \geqq p_\nu$, the conditions (3.1), (3.2) *are satisfied, then there exists a comonic matrix polynomial $L(\lambda)$ of degree l such that $L(\lambda)$ has a canonical pair of the form $((X_F X_\infty), J_F^{-1} \oplus J_\infty)$, where J_∞ is a nilpotent Jordan matrix with Jordan blocks of sizes $p_1, \cdots, p_\nu$.*

PROOF. The proof of this theorem is based heavily on theorem 0.1 [4]. We first prove the necessity of conditions (3.1) and (3.2). Let $\tilde{L}(\lambda) = \lambda^l L(\lambda^{-1})$ be the monic polynomial of degree l, and let (X, J) be its standard pair. In view of theorem 3.1 [3], (X_F, J_F^{-1}) is similar to the part of (X, J) corresponding to the non-zero eigenvalues of $\tilde{L}(\lambda)$, and (X_∞, J_∞) is similar to the part of (X, J) corresponding to the eigenvalue 0. Applying theorem 0.1 [4] we obtain inequalities (3.1) and (3.2).

Conversely, suppose $p_1 \geqq \cdots \geqq p_\nu$ are given such that (3.1) and (3.2) hold. Applying corollary 3.1 [4] we find an admissible pair (X_∞, J_∞) such that $\mathrm{co}(X_F J_F^{-j+1}\, X_\infty J_\infty^{j-1})_{j=1}^{l}$ is invertible, where J_∞ is a nilpotent Jordan matrix with the dimensions of its Jordan blocks equal to $p_1, \cdots, p_\nu$. Let $\tilde{L}(\lambda)$ be a monic polynomial of degree l with standard pair $((X_F X_\infty), J_F^{-1} \oplus J_\infty)$. In view of theorem 3.1 [3], the comonic polynomial $L(\lambda) = \lambda^l \tilde{L}(\lambda^{-1})$ is the desired one.

Now let $L(\lambda)$ be a matrix polynomial with discrete spectrum and let r be the degree of $\det L(\lambda)$. An admissible pair (X, J) of eigenvectors and generalized eigenvectors of $L(\lambda)$ consists of an $n \times r$ matrix X, the columns of which form a canonical set of eigenvectors and generalized eigenvectors of $L(\lambda)$ (in finite points only), and of the corresponding $r \times r$ Jordan matrix J.

THEOREM 3.2. *Let $L(\lambda)$ be a matrix polynomial of degree l and let $\alpha \in \mathbf{C} \setminus \sigma(L)$. Then there exists a matrix polynomial $L_1(\lambda)$ with $\sigma(L_1) = \{\alpha\}$ such that $L_1(\lambda)L(\lambda)$ is a monic polynomial of degree l. The partial multiplicities $p_1 \geqq \cdots \geqq p_\nu$ of $L_1(\lambda)$ satisfy* (3.1) *and* (3.2), *where $s_0, s_1, \cdots$ are the s-indices of an admissible pair (X_F, J_F) of eigenvectors and generalized eigenvectors of $L(\lambda)$, and r is the degree of* $\det L(\lambda)$.

Conversely, if (3.1) *and* (3.2) *are satisfied for some positive integers $p_1 \geqq \cdots \geqq p_\nu$, then there exists a matrix polynomial $L_1(\lambda)$ with $\sigma(L_1) = \{\alpha\}$ such that $p_1, \cdots, p_\nu$ are its partial multiplicities and $L_1(\lambda)L(\lambda)$ is a monic polynomial of degree l.*

This theorem will be proved in the next section.

§4. The structure of right divisors

Let $L(\lambda)$ be a comonic matrix polynomial, and let (X_F, J_F^{-1}) be its finite canonical pair. In this section, we establish the connection between right comonic divisors of $L(\lambda)$ and J_F-invariant subspaces.

THEOREM 4.1. *Let Λ be a J_F-invariant subspace, and let* $m_0 = \min\{m \geqq 1 \mid (X_F \mid \Lambda, J_F^{-1} \mid \Lambda)$ is m-independent$\}$. *Then a comonic polynomial*

$$L_\Lambda(\lambda) = I - (X_F \mid \Lambda) \cdot (J_F^{-m_0} \mid \Lambda) \cdot (V_1 \lambda^{m_0} + \cdots + V_{m_0} \lambda),$$

where $(V_1 \cdots V_{m_0})$ is a special left inverse of $\operatorname{col}((X_F \mid \Lambda)(J_F \mid \Lambda)^{-j+1})_{j=1}^{m_0}$, *is a right divisor of $L(\lambda)$. Conversely, for every comonic right divisor $L_1(\lambda)$ of $L(\lambda)$ there exists a unique J_F-invariant subspace Λ such that $L_1(\lambda) = U(\lambda)L_\Lambda(\lambda)$, where $U(\lambda)$ is an everywhere invertible matrix polynomial.*

PROOF. We first prove that $L_\Lambda(\lambda)$ is a right divisor of $L(\lambda)$ for any J_F-invariant subspace Λ.

Let l_1 and l_2 be positive integers which satisfy the following condition:

(i) $$\operatorname{Ker} \operatorname{col}((X_F \mid \Lambda) \cdot (J_F \mid \Lambda)^{-j+1})_{j=1}^{l_1} = \{0\};$$

(ii) $$r - nl_2 \leqq \dim \Lambda,$$

where r is the degree of $\det L(\lambda)$;

(iii) $$l \overset{\text{def}}{=} l_1 + l_2 \geqq \operatorname{degree}(L(\lambda)).$$

The pair (X_F, J_F^{-1}) is l-independent (in view of (iii)). Using corollary 3.1 of [4], it is easy to see that there exists an admissible pair (X_0, J_0) with nilpotent Jordan matrix J_0 such that $\operatorname{col}(X_F J_F^{-j+1} X_0 J_0^{j-1})_{j=1}^{l}$ is square and invertible, and for some J_0-invariant subspace Θ,

$$\operatorname{col}((X_F \mid \Lambda)(J_F \mid \Lambda)^{-j+1}(X_0 \mid \Theta)(J_0 \mid \Theta)^{j-1})_{j=1}^{l_1}$$

is square and invertible. For example, one can choose J_0 as a single Jordan block of size $nl - r$. Let $\tilde{L}(\lambda)$ be the monic polynomial of degree l defined by the standard pair $((X_F X_0), J_F^{-1} \oplus J_0)$. Then (theorem 8 of [2]) there exists a right monic divisor $\tilde{L}(\lambda)$ of degree l_1 of $\tilde{L}(\lambda)$, corresponding to the subspace $\Lambda + \Theta$: $\tilde{L}(\lambda) = \tilde{L}_2(\lambda)\tilde{L}_1(\lambda)$, where the monic polynomial $\tilde{L}_2(\lambda)$ is of degree l_2. Define comonic polynomials $M(\lambda) = \lambda^l \tilde{L}(\lambda^{-1})$, $M_1(\lambda) = \lambda^{l_1}\tilde{L}_1(\lambda^{-1})$, and $M_2(\lambda) = \lambda^{l_2}\tilde{L}_2(\lambda^{-1})$. Then

$$M(\lambda) = M_2(\lambda) M_1(\lambda), \tag{4.1}$$

and the finite canonical pairs of $M(\lambda)$ and $M_1(\lambda)$ are similar to (X_F, J_F^{-1}) and $(X_F \mid \Lambda, J_F^{-1} \mid \Lambda)$ respectively. Now from Theorem 2.1 it follows that $(X_F \mid \Lambda, J_F^{-1} \mid \Lambda)$ is also similar to a finite canonical pair of $L_\Lambda(\lambda)$. Using (4.1) and theorem 5.1 of [3], we see that $L_\Lambda(\lambda)$ is indeed a right divisor of $L(\lambda)$. These arguments also serve to prove the converse statement of the theorem.

The comonic right divisors of $L(\lambda)$ corresponding to a J_F-invariant subspace Λ, will be called the divisors *generated* by Λ, and Λ is their *generating subspace.*

Let us now make some remarks connected with Theorem 4.1. The equation (4.1) shows that there exists a divisor $L_1(\lambda)$ generated by Λ of degree $\leqq l_1$ such that the degree of $L(\lambda)\, L_1^{-1}(\lambda)$ is $\leqq l_2$, whenever the conditions (i), (ii), and (iii) hold. Two divisors are generated by Λ if and only if they have the same Jordan chains corresponding to the same eigenvalues. An admissible pair of eigenvectors and generalized eigenvectors for any divisor generated by Λ is similar to the restriction $(X_F \mid \Lambda, J_F \mid \Lambda)$. It is possible to extend Theorem 4.1 to the case where $L(\lambda)$ has a discrete spectrum (by shifting the argument $\lambda \to \lambda + a$).

Using Theorem 2.1 and the proof of Theorem 4.1, we obtain the following corollary.

COROLLARY 4.1. *The minimal degree of a right comonic divisor of* $L(\lambda)$ *generated by* Λ *is* $\min\{m \geqq 1 \mid \operatorname{Ker}[\operatorname{col}((X_F \mid \Lambda)\,(J_F \mid \Lambda)^{-j+1})_{j=1}^{m}] = \{0\}\}$ *if* $\Lambda \neq \{0\}$, *and zero if* $\Lambda = \{0\}$.

It is possible to formulate conditions for the divisibility of matrix polynomials in terms of their spectral data. The following theorem gives one such possible formulation.

THEOREM 4.2. *Let $M_1(\lambda)$ and $M_2(\lambda)$ be matrix polynomials with discrete spectrum. Then $M_1(\lambda)$ is a right divisor of $M_2(\lambda)$ if and only if every Jordan chain of $M_1(\lambda)$ is at the same time a Jordan chain of $M_2(\lambda)$ corresponding to the same eigenvalue.*

PROOF. If $M_1(\lambda)$ is a right divisor of $M_2(\lambda)$, then one can verify directly that every Jordan chain of $M_1(\lambda)$ is a Jordan chain of $M_2(\lambda)$ corresponding to the same eigenvalue.

Suppose now that every Jordan chain of $M_1(\lambda)$ is a Jordan chain of $M_2(\lambda)$ corresponding to the same eigenvalue. Without loss of generality we can assume that $M_1(\lambda)$ and $M_2(\lambda)$ are comonic. Let (X_{1F}, J_{1F}) and (X_{2F}, J_{2F}) be admissible pairs of eigenvectors and generalized eigenvectors of $M_1(\lambda)$ and $M_2(\lambda)$ respectively. Then the conditions of the theorem imply that (X_{1F}, J_{1F}) is similar to the restriction $(X_{2F} \mid \Lambda, J_{2F} \mid \Lambda)$, where Λ is a J_{2F}-invariant subspace.

Indeed, without loss of generality, we can suppose that $M_1(\lambda)$ and $M_2(\lambda)$ have the same degree. From the linearization (1) of [3], it follows that every Jordan chain of $M_1(\lambda)$ is a Jordan chain of $M_2(\lambda)$ corresponding to the same eigenvalue if and only if the same is true for $I - \lambda R_1$ and $I - \lambda R_2$, where R_1 and R_2 are the companion matrices of $M_1(\lambda)$ and $M_2(\lambda)$ respectively. Moveover, we can restrict ourselves to the case $M_1(\lambda) = I - \lambda R_1$, and $M_2(\lambda) = I - \lambda R_2$. The assertion should now be clear.

Let $N(\lambda)$ be a right divisor of $M_2(\lambda)$ which is generated by Λ. Then $N(\lambda)$ and $M_1(\Lambda)$ have the same Jordan chains and therefore (see theorem 5.1 of [3]) $N(\lambda) = U(\lambda) \cdot M_1(\lambda)$ for some everywhere invertible matrix polynomial $U(\lambda)$. Thus $M_1(\lambda)$ is a right divisor of $M_2(\lambda)$.

PROOF OF THEOREM 3.2. We first prove the existence of $L_1(\lambda)$. Note that the s-indices of $(X_F, J_F - aI)$ do not depend on the choice of $a \in C$. Therefore, without loss of generality, we can suppose that J_F is invertible and that $L(\lambda)$ is comonic.

Let l be the degree of $L(\lambda)$. In view of corollary 3.1 of [4], there exists an admissible pair (X_0, J_0), where J_0 is a Jordan matrix with $\sigma(J_0) = \{\alpha\}$, such that the matrix $\operatorname{col}(X_F J_F^{j-1} X_0 J_0^{j-1})_{j=1}^{l}$ is square and invertible. Let $L_0(\lambda)$ be the monic polynomial of degree l the standard pair of which is $((X_F X_0), J_F \oplus J_0)$.

In view of Theorem 4.2, $L_0(\lambda) = M(\lambda) L(\lambda)$ for some matrix polynomial

$M(\lambda)$. Since the multiplicity of every eigenvalue (except α) of $L(\lambda)$ and $L_0(\lambda)$ is the same, $\sigma(M)=\{\alpha\}$. From corollary 3.1 of [4], it should be clear (upon repeating the above arguments) that the converse statement of Theorem 3.2 holds true also.

§5. The quotient and classes of divisors

Let $L(\lambda)$ be a comonic matrix polynomial of degree $\leqq l$ and with degree $(\det L(\lambda))=r$. In this section, we study the degrees of the right divisors $L_1(\lambda)$ of $L(\lambda)$ together with the degrees of the quotients $L(\lambda)L_1^{-1}(\lambda)$.

Let (X_F, J_F^{-1}) be a finite canonical pair of $L(\lambda)$. Let Λ be a J_F-invariant subspace of $\mathbf{C}_r$. As was shown in §4, the minimal possible degree of a right divisor of $L(\lambda)$ which is generated by Λ is

$$m_1=\min\{m\geqq 1 \mid \operatorname{Ker}[\operatorname{col}(X_F\mid\Lambda)\cdot(J_F\mid\Lambda)^{i-1})_{i=1}^{m}]=\{0\}\}.$$

The minimal degree of the quotient $L_2(\lambda)=L(\lambda)L_1^{-1}(\lambda)$, where $L_1(\lambda)$ runs over the set of all right divisors of $L(\lambda)$ of degree m_1, which are generated by Λ, is

$$m_2=[\max(\text{degree } L(\lambda)-m_1,(r-\dim\Lambda)\cdot n^{-1})],$$

where $[x]$ denotes the smallest integer which is larger than or equal to x. Indeed, m_2 is the smallest integer such that the conditions (ii) and (iii) of §4 hold. It follows from the proof of Theorem 4.1 that, among all the right divisors of degree m, which are generated by Λ, there exists a matrix polynomial $L_1(\lambda)$ such that the degree of $L(\lambda)L_1^{-1}(\lambda)$ is $\leqq m_2$. From the same proof it is also clear that m_2 is minimal.

Let $\operatorname{Div}(l_1,l_2)$ be the set of all the right divisors of $L(\lambda)$ of degree $\leqq l_1$ such that the degree of the quotient is $\leqq l_2$. We say that two divisors $M_1(\lambda)$ and $M_2(\lambda)$ of $L(\lambda)$ *belong to the same class* if $M_1(\lambda)=U(\lambda)M_2(\lambda)$ for some everywhere invertible matrix polynomial $U(\lambda)$. From the results of §4, it follows that $M_1(\lambda)$ and $M_2(\lambda)$ belong to the same class if and only if their generating subspaces coincide (by definition the generating subspace of $M_1(\lambda)$ is the generating subspace of the comonic divisor $M_1^{-1}(0)M_1(\lambda)$). In [3], the following problem was mentioned: when does the set $\operatorname{Div}(l_1,l_2)$ contain a representative from every class of divisors? A partial answer was given: if l_1, l_2 are big enough, then $\operatorname{Div}(l_1,l_2)$ contains a representative from every class of divisors. In this section, we give a complete answer.

Let ν be the smallest positive integer such that $\operatorname{Ker}[\operatorname{col}(X_F J_F^{i-1})_{i=1}^{\nu}]=\{0\}$. Then $\operatorname{Div}(\nu,[rn^{-1}])$ contains a representation from every class of right divisors of $L(\lambda)$.

Conversely, if for some l_1, l_2 the set $\mathrm{Div}(l_1, l_2)$ contains a representative from every class of right divisors of $L(\lambda)$, then $l_1 \geqq \nu$, $l_2 \geqq [rn^{-1}]$.

To check these statements, choose $l_1 = \nu$, $l \geqq \nu + [rn^{-1}]$ in conditions (i), (ii) and (iii) of §4. Then these conditions are satisfied for every J_F-invariant subspace Λ. Thus every J_F-invariant subspace is a generating subspace of some divisor from $\mathrm{Div}(\nu, [rn^{-1}])$. Since any two divisors generated by the same subspace, belong to the same class, we see that $\mathrm{Div}(\nu, [rn^{-1}])$ contains a representative from each class.

Suppose now that $\mathrm{Div}(l_1, l_2)$ contains a representative from each class of divisors. Then conditions (i) and (ii) hold for every J_F-invariant subspace Λ. In particular, taking $\Lambda = 0$ in (ii) and $\Lambda = \mathbf{C}_r$ in (i), we obtain $l_2 \geqq rn^{-1}$ and $l_1 \geqq \nu$.

EXAMPLE 5.1. To illustrate the theory consider the following example. Let

$$L(\lambda) = \begin{pmatrix} (\lambda+1)^3 & \lambda \\ 0 & (\lambda-1)^2 \end{pmatrix}.$$

Then a finite canonical pair (X_F, J_F^{-1}) of $L(\lambda)$ is given by the formulas

$$X_F = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -8 & -4 \end{pmatrix}, \quad J_F^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix} \oplus \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

We compute the divisors generated by the J_F-invariant subspace Λ spanned by the vectors $(1\,0\,0\,0\,0)^T$, $(0\,1\,0\,0\,0)^T$, $(0\,0\,0\,1\,0)^T$. We have:

$$X_F \mid \Lambda = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & -8 \end{pmatrix}, \quad J_F^{-1} \mid \Lambda = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The columns of

$$\begin{bmatrix} X_F \mid \Lambda \\ X_F \mid \Lambda \cdot J_F^{-1} \mid \Lambda \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -8 \\ -1 & -1 & 1 \\ 0 & 0 & -8 \end{bmatrix}$$

are independent, and a special left inverse is

$$(V_1 V_2) = \begin{bmatrix} 1 & 0 & 0 & \frac{1}{8} \\ -1 & 0 & -1 & -\frac{2}{8} \\ 0 & 0 & 0 & -\frac{1}{8} \end{bmatrix}.$$

Therefore,

$$I-(X_F\,|\,\Lambda)(J_F^{-2}\,|\,\Lambda)(V_1\lambda^2+V_2\lambda)=\begin{pmatrix}(\lambda+1)^2 & \lambda/2\\ 0 & \lambda-1\end{pmatrix}$$

and the general form of a right comonic divisor generated by Λ is

$$U(\lambda)\cdot\begin{pmatrix}(\lambda+1)^2 & \lambda/2\\ 0 & \lambda-1\end{pmatrix},$$

where $U(\lambda)$ is matrix polynomial with $U(0)=I$ and $\det U(\lambda)\equiv 1$.

The set Div (3,3) contains a representative from every class of right divisors of $L(\lambda)$ (in our case $t=5$, $n=2$, $\nu=3$).

§6. Spectral divisors

Let $L(\lambda)$ be an $n\times n$ matrix polynomial with discrete spectrum of degree $\leqq l$. A point $\lambda\in\mathbf{C}$ is *regular* for $L(\lambda)$ if $\lambda\not\in\sigma(L)$. Let Γ be a contour in the complex plane consisting only of regular points for $L(\lambda)$. A right (left) divisor $M(\lambda)$ of $L(\lambda)$ is called Γ-*spectral* if $\sigma(M)$ is inside Γ and $\sigma(LM^{-1})$ ($\sigma(M^{-1}L)$) is outside Γ.

For every contour Γ, consisting only of regular points for $L(\lambda)$, there exist a Γ-spectral right divisor and a Γ-spectral left divisor. Indeed, we can suppose that $L(\lambda)$ is a comonic polynomial. Let (X_F,J_F^{-1}) be its finite canonical pair. Let

$$\Lambda=\operatorname{Im}\frac{1}{2\pi i}\int_\Gamma(\lambda I-J_F)^{-1}\,d\lambda.$$

Then any right comonic divisor $M(\lambda)$ of $L(\lambda)$ which is generated by Λ is Γ-spectral (because the number of roots of $\det M(\lambda)$ is equal to the number of roots of $\det L(\lambda)$ inside Γ). Note that the converse is also true, i.e. if $M(\lambda)$ is a Γ-spectral right comonic divisor of $L(\lambda)$, then $M(\lambda)$ is generated by Λ. The existence of a Γ-spectral left divisor of $L(\lambda)$ can be proved analogously.

THEOREM 6.1. *Let Γ be a contour consisting only of regular points of $L(\lambda)$, such that $L(\lambda)$ has eigenvalues inside Γ. Let $a\not\in\sigma(L)$ be a complex number lying outside Γ. Then the smallest possible degree of a Γ-spectral right divisor of $L(\lambda)$ is equal to the smallest integer $j\geqq 1$ for which*

$$\operatorname{rank}\frac{1}{2\pi i}\int_\Gamma \operatorname{col}(\Omega_k\otimes L^{-1}(\lambda))_{k=1}^{j}\,d\lambda$$

is maximal, where

$$\Omega_k = ((\lambda - a)^{l-k-1}(\lambda - a)^{l-k-2} \cdots (\lambda - a)^{-k}).$$

PROOF. Let $\tilde{L}(\lambda)$ be the monic polynomial of degree l such that $L^{-1}(a) \cdot L(\lambda) = (\lambda - a)^l \cdot \tilde{L}((\lambda - a)^{-1})$. Let $\tilde{\Gamma} = \{(\lambda - a)^{-1} \mid \lambda \in \Gamma\}$. Then for a given integer j,

$$\int_\Gamma (\lambda - a)^j L^{-1}(\lambda) d\lambda = -\int_{\tilde{\Gamma}} \lambda^{-j+l-2} \tilde{L}^{-1}(\lambda) d\lambda \cdot L^{-1}(a),$$

and therefore

$$\operatorname{rank} \operatorname{col}\left(\frac{1}{2\pi i}\int_\Gamma \Omega_k \otimes L^{-1}(\lambda)\, d\lambda\right)_{k=1}^{j} = \operatorname{rank} \operatorname{col}\left(\frac{1}{2\pi i}\int_{\tilde{\Gamma}} \Phi_k \otimes \tilde{L}^{-1}(\lambda)\, d\lambda\right)_{k=1}^{j},$$

where $\Phi_k = (\lambda^{k-1}\lambda^k \cdots \lambda^{l+k-2})$ for $k = 1, 2, \cdots$. Let (Q, T, R) be a standard triple of $\tilde{L}(\lambda)$. Then (cf. the proof of theorem 2.1 [1])

$$\operatorname{col}\left(\frac{1}{2\pi i}\int_{\tilde{\Gamma}} \Phi_k \otimes \tilde{L}^{-1}(\lambda) d\lambda\right)_{k=1}^{j} = \operatorname{col}(Q_1 T_1)_{k=0}^{j-1} \cdot (R_1 \; T_1 R_1 \cdots T_1^{l-1} R_1),$$

where (Q_1, T_1, R_1) is the part of (Q, T, R) corresponding to the eigenvalues of $\tilde{L}(\lambda)$ lying inside $\tilde{\Gamma}$.

Let s be the number of zeroes of $\det L(\lambda)$ inside Γ (counting multiplicities). Since zero is outside $\tilde{\Gamma}$, $\det \tilde{L}(\lambda)$ has exactly s zeroes inside $\tilde{\Gamma}$. Therefore,

$$\Lambda = \operatorname{Im}(R_1 \; T_1 R_1 \cdots T_1^{l-1} R_1) = \operatorname{Im}\left[\frac{1}{2\pi i}\int_{\tilde{\Gamma}} (\lambda I - T)^{-1} d\lambda\right]$$

is an s-dimensional subspace, and

$$\operatorname{rank} \frac{1}{2\pi i}\int_\Gamma \operatorname{col}(\Omega_k \otimes L^{-1}(\lambda))_{k=1}^{j}\, d\lambda$$

is maximal if and only if

$$\operatorname{Ker} \operatorname{col}(Q_1 T_1^k)_{k=0}^{j-1} = \operatorname{Ker} \operatorname{col}(Q(T \mid \Lambda)^k)_{k=0}^{j-1} = \{0\}.$$

In view of Corollary 4.1, the last condition is equivalent to the existence of a Γ-spectral right divisor of $L(\lambda)$ of degree j. Q.E.D.

Taking transposes, we obtain the following result.

THEOREM 6.2. *Under the conditions of Theorem 6.1, the smallest possible degree of a Γ-spectral left divisor of $L(\lambda)$ is equal to the smallest integer $j \geqq 1$, for which*

$$\operatorname{rank} \frac{1}{2\pi i} \int_{\Gamma} (\Psi_1 \otimes L^{-1}(\lambda) \cdots \Psi_j \otimes L^{-1}(\lambda)) d\lambda$$

is maximal, where $\Psi_k = \operatorname{col}((\lambda - a)^{l-k-j})_{j=1}^{l}$.

Note that the condition that a be outside Γ is crucial in Theorems 6.1 and 6.2.

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